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Stability characteristics for flows of the vortex-sheet type

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Interfacial conditions for a cylindrical vortex sheet or a cylindrical fluid layer with radius-dependent density, velocity and magnetic fields are derived for isentropic compressible swirling flows subjected to arbitrary disturbances. Surface tension is included for possible immiscible fluids. These conditions are valid for both spatially and temporally growing waves and for flow profiles with or without discontinuities. The deformation of the sheet or the layer affects the flow in two ways: perturbing the total pressure field and disturbing the centrifugal force field created by the azimuthal components of the velocity and the magnetic flux. The latter seems to be straightforward, but is easily overlooked as in some of the previous analyses. We will show that failure to consider such a perturbation to a stable centrifugal force field will lead to the improper destabilization of certain modes with smaller axial and azimuthal wavenumbers.

The interfacial conditions and their corresponding stability characteristics are further examined for a general class of incompressible flows subject to temporal perturbations. Unlike the single role of destabilization played by the velocity in two-dimensional stratified flows or axisymmetric jet flows, the rotating velocity in vortex motions plays a dual role in flow stability: the angular-velocity gradient generates tangential shear, and the angular velocity itself creates a centrifugal force field. While the former always destabilizes the flow, the latter can either stabilize or destabilize the flow depending on whether the resultant force is centrifugally stable or unstable. These characteristics are demonstrated by examining three general types of perturbations.

1. Introduction

In a paper by Rotunno (1978), the uncertainties of the stability analysis for an incompressible cylindrical vortex sheet in a homogeneous inviscid fluid were again discussed. Using a potential formulation for both vorticity-free regions inside and outside the vortex sheet, he resolved the inconsistency in an earlier analysis by Michalke & Timme (1967) and recovered the first two otherwise stable modes. A later analysis by Leibovich (1969) on stability of inviscid rotating coaxial jets in stratified fluids revealed similar uncertainties arising from the perturbations to flows with discontinuity profiles. The author found that the stability characteristics uncovered were anomalous and concluded that errors might result if a thin but stable layer was replaced by a vortex sheet. The purpose of this paper is to investigate some uncertainties of the stability analysis for flows with discontinuity profiles and to present an overall view for flows of this type. The flow to be considered is isentropic and has general radius-dependent profiles for the density, velocity and magnetic fields. All dissipative effects are disregarded. Surface tension is also included for the case of immiscible fluids.

Even though many criteria have been derived (e.g. Howard & Gupta 1962; Fung & Kurzweg 1975; Lalas 1975; Fung 1982) for general vortex flows to provide us with some upper-bound information on stability or instability, criteria for flows of this kind do not yield sufficient knowledge of instabilities, if any, for a given flow profile. Solutions to the governing stability equations must be obtained before the detailed instability characteristics for a particular flow profile can be observed. Unfortunately, analytical solutions in terms of well-known functions for general vortex flows are very difficult to obtain except for a few broken-line profiles. Matching the solutions at the common boundary between two flow regions therefore becomes the trick for the analysis of this type (e.g. Michalke & Timme 1967; Leibovich 1969; Lessen, Deshpande & Hadji-Ohanes 1973). In matching those broken-line profiles, appropriate interfacial conditions must be used.

In the present analysis, we will show from the derivation of the interfacial conditions that perturbations to the flow disturb both the pressure field and the centrifugal force field. The latter, created by the fluid rotation and the azimuthal magnetic field, stabilizes or destabilizes the flow depending on whether the force field generated is centrifugally stable or unstable. The perturbation to the centrifugal force field is essential to the flow characteristics especially when rapid changes of flow quantities exist within a thin layer of fluid. Such a perturbation to the centrifugal force field seems to be straightforward, but is sometimes easily overlooked.

When discontinuities of flow quantities exist in a cylindrical interface, instabilities are likely to occur because of the sharp velocity gradient and any unbalanced centrifugal forces present at the interface. The rotation of fluid particles plays a dual role in flow characteristics. Though the angular velocity gradient generates shear effects which always destabilize the flow, the angular velocity itself induces centrifugal forces which can either stabilize or destabilize the flows, depending on whether the force field induced is centrifugally stable or unstable. This phenomenon will be demonstrated by examining a general class of vortex-sheet-type flows. The stabilizing or destabilizing effect of the centrifugal force field will be revealed by the perturbation of the field at the interface. A centrifugally stable force field created by the rotation and the azimuthal magnetic field may not always offset the shear instability of the vortex sheet, but certainly will stabilize disturbances corresponding to longer wavelengths.

2. Governing equations for normal modes

Consider a swirling flow with a velocity $\boldsymbol{U}$ to be confined within the annular region (r, θ, z) between two rigid, infinite and coaxial cylinders in the presence of a magnetic field $\boldsymbol{H}$. The fluid having an inhomogeneous density ρ^* is assumed to be compressible but non-heat-conducting. In the absence of gravitational forces and dissipation effects due to viscosity, magnetic resistivity and thermal diffusivity, the governing equations for the isentropic motion of the flow are

$$\rho^* \frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} = \nabla Q + \frac{\mu}{4\pi}(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}, \tag{1}$$

$$\frac{\mathrm{D}\rho^*}{\mathrm{D}t} + \rho^* \nabla\cdot\boldsymbol{U} = 0, \tag{2}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = \nabla \times (\boldsymbol{U}\times\boldsymbol{H}), \tag{3}$$

$$\nabla \cdot \boldsymbol{H} = 0, \tag{4}$$

$$\frac{\mathrm{D}P}{\mathrm{D}t} = a^2 \frac{\mathrm{D}\rho^*}{\mathrm{D}t}, \tag{5}$$

where μ denotes the magnetic permeability. The total pressure Q (including the magnetic pressure) is related to the hydrodynamic pressure P as follows:

$$Q = P + \frac{\mu}{4\pi}|\boldsymbol{H}|^2. \tag{6}$$

The velocity of sound for an isentropic process is given by

$$a^2 = \left(\frac{\partial P}{\partial \rho^*}\right)_{\text{isentropic}}.$$

For given velocity and magnetic fields only one of the thermodynamic variables can be independently prescribed. The boundary conditions for the system governed by (1)–(5) are those of perfectly conducting rigid walls.

The flow to be considered has a steady-state, radius-dependent profile: $V_0(r)$ is the azimuthal velocity, $W_0(r)$ the axial velocity, $H_\theta(r)$ the azimuthal magnetic field, $H_z(r)$ the axial magnetic field, $Q_0(r)$ the total pressure, $\rho_0(r)$ the density and $a_0(r)$ the sound speed. Let the flow be perturbed as follows:

$$\left.\begin{aligned} \boldsymbol{U} &= \boldsymbol{U}[\hat{u}, V_0(r)+\hat{v}, W_0(r)+\hat{w}], \\ \boldsymbol{H} &= \boldsymbol{H}[\hat{h}_r, H_\theta(r)+\hat{h}_\theta, H_z(r)+\hat{h}_z], \\ Q &= Q_0(r)+\hat{q}, \\ \rho^* &= \rho_0(r)+\hat{\rho}, \\ a &= a_0(r)+\hat{a}. \end{aligned}\right\} \tag{7}$$

We further introduce the periodic solutions

$$\hat{\phi} = \phi(r)\exp[\mathrm{i}(kz+m\theta-\omega t)], \tag{8}$$

such that the azimuthal wavenumber m is an integer, and the axial wavenumber k and the circular frequency ω are both complex in order to admit solutions for both spatially and temporally growing disturbances. Within the framework of the normal-mode approach, the linearized equations for the flow described by (1)–(5), subject to small perturbations, are given as follows:

$$\left(1-\frac{N_\mathrm{A}^2}{N^2}\right)\mathrm{D}^*\left(\frac{u}{N}\right) - \frac{2mV_0}{Nr^2}\left(1-\frac{N_\mathrm{A}}{N}\frac{V_\mathrm{A}}{V_0}\right)\left(\frac{u}{N}\right) - \frac{\mathrm{i}}{\rho_0}\frac{k^2+m^2/r^2}{N^2}q = -\frac{G}{a_0^2}, \tag{9}$$

$$\begin{aligned} &\left\{\left(1-\frac{N_\mathrm{A}^2}{N^2}\right)\left[(N^2-\Phi)-(N_\mathrm{A}^2-\Psi_\mathrm{A})+\frac{T}{\rho_0 R^2}(1-m^2-k^2R^2)\,\delta(r-R)\right]\right. \\ &\quad \left. -4\frac{V_0^2}{r^2}\frac{N_\mathrm{A}^2}{N^2}\left[\left(1-\frac{V_\mathrm{A}}{V_0}\right)^2+2\frac{V_\mathrm{A}}{V_0}\left(1-\frac{N}{N_\mathrm{A}}\right)\right]\right\}\left(\frac{u}{N}\right) \\ &\quad -\frac{\mathrm{i}}{\rho_0}\left[\left(1-\frac{N_\mathrm{A}^2}{N^2}\right)\mathrm{D}q+\frac{2mV_0}{Nr^2}\left(1-\frac{V_\mathrm{A}}{V_0}\frac{N_\mathrm{A}}{N}\right)q\right] \\ &\qquad = -\frac{V_0^2}{r}\left[\left(\frac{N_\mathrm{A}}{N}-\frac{V_\mathrm{A}}{V_0}\right)^2-\left(1-\frac{V_\mathrm{A}^2}{V_0^2}\right)\right]\frac{G}{a_0^2}, \end{aligned} \tag{10}$$

where

$$G = (V_A^2 + W_A^2)\,D^*\left(\frac{u}{N}\right) - \frac{N^2}{N^2-N_A^2}\left\{\frac{V_0^2}{r}\left[\left(\frac{N_A}{N}-\frac{V_A}{V_0}\right)^2 - \left(1-\frac{V_A^2}{V_0^2}\right)\right]\right.$$

$$\left. + \frac{V_A^2+W_A^2}{r}\left[\frac{2mV_0}{Nr}\left(1-\frac{N_A}{N}\frac{V_A}{V_0}\right)\right]\right\}\left(\frac{u}{N}\right)$$

$$+ \frac{\mathrm{i}}{\rho_0}\frac{N^2}{(N^2-N_A^2)}\left[1-(V_A^2+W_A^2)\frac{k^2+m^2/r^2}{N^2}\right]q,$$

and $N = kW_0 + mV_0/r - \omega$ is the Doppler-shifted frequency, $N_A = kW_A + mV_A/r$ the Alfvén frequency, $V_A = (\mu/4\pi\rho_0)^{\frac{1}{2}} H_\theta$ the azimuthal Alfvén velocity, $W_A = (\mu/4\pi\rho_0)^{\frac{1}{2}} H_z$ the axial Alfvén velocity, $D = d/dr$, $D^* = D + 1/r$ and $D_* = D - 1/r$. The surface-tension effect for possible immiscible fluids is introduced in (10), with T representing the surface-tension coefficient, $\delta(r-R)$ the Dirac delta function, and R the radial position for a cylindrical vortex sheet or a cylindrical fluid layer. The Rayleigh–Synge discriminant is defined as

$$\Phi = \frac{D[\rho_0(rV_0)^2]}{\rho_0 r^3}$$

and the Alfvén discriminant as

$$\Psi_A = \frac{r}{\rho_0} D\left[\rho_0\left(\frac{V_A}{r}\right)^2\right].$$

The boundary condition for (9) and (10) is that u vanishes at the inner and outer boundaries. The two discriminants play a crucial role in the flow stability. For incompressible flows subject to axisymmetric disturbances and with all the axial influence in the velocity and magnetic field suppressed, they constitute a necessary and sufficient condition for stability, i.e.

$$\Phi - \Psi_A \geqslant 0. \tag{11}$$

The above criterion can be easily obtained from (9) and (10), and will be called the generalized Michael condition (Michael 1954).

Assuming that a vortex sheet or a fluid layer in its steady state is located at $r = R$ with possible discontinuities in all components of the density, velocity and magnetic fields, one can integrate (9) and (10) across the vortex sheet to obtain the kinematic and dynamic interfacial conditions:

$$\left\langle \frac{u}{N} \right\rangle = 0, \tag{12}$$

$$\langle q \rangle - \mathrm{i}\left(\frac{u}{N}\right)_R \left[\left\langle \rho_0\left(\frac{V_0^2}{r} - \frac{V_A^2}{r}\right)\right\rangle + \frac{T}{R^2}(k^2R^2 + m^2 - 1)\right] = 0, \tag{13}$$

where $\langle \phi \rangle = \phi(R_{+0}) - \phi(R_{-0})$ denotes a possible jump condition at the interface. Equation (12) simply states the well-known fact that the Lagrangian displacement should be continuous across the interface. Equation (13) points out that the dynamic interfacial condition should include not only the perturbations to the surface tension and the total pressure field, but also the perturbations to the centrifugal force field resulting from the velocity and Alfvén waves in the azimuthal direction. The jump condition arising from the latter perturbations is the outcome of integrating the Rayleigh–Synge and the Alfvén discriminants across the interface. This outcome, supported by the analysis to be given later, implies that the generalized Michael condition is a differential representation of a stable centrifugal force field. Compressibility effects do not explicitly enter into either of the interfacial conditions. This characteristic can be seen from an alternative derivation of both conditions in the following section.

3. Physical considerations

Assume that the cylindrical vortex sheet located at the radial position R is disturbed such that the deformed interface is prescribed by

$$r = R+\hat{\eta}(r, \theta, z; t), \tag{14}$$

where $R \gg \hat{\eta}$. Taking the total derivative of (14) and assuming periodic solutions yield

$$\eta(r) = -\mathrm{i}\frac{u(r)}{kW_0+mV_0/r-\omega}. \tag{15}$$

Equation (12) immediately follows if no gap is allowed to exist at the disturbed interface.

The dynamic interfacial condition can also be obtained by examining the Euler equation of motion. In the presence of surface-tension effects, the steady-state form of (1) in the radial direction yields

$$\mathrm{D}Q_0 = \rho_0\left(\frac{V_0^2}{r}-\frac{V_A^2}{r}\right)-\frac{T}{R}\delta(r-R). \tag{16}$$

The steady-state total pressures inside and outside the vortex sheet are respectively

$$Q_{01}(r) = \int_{R_1}^{r}\rho_{01}\left(\frac{V_{01}^2}{\xi}-\frac{V_{A1}^2}{\xi}\right)\mathrm{d}\xi \quad (R_1 \leqslant r < R), \tag{17}$$

$$Q_{02}(r) = \int_{R_1}^{R}\rho_{01}\left(\frac{V_{01}^2}{\xi}-\frac{V_{A1}^2}{\xi}\right)\mathrm{d}\xi+\int_{R}^{r}\rho_{02}\left(\frac{V_{02}^2}{\xi}-\frac{V_{A2}^2}{\xi}\right)\mathrm{d}\xi-\frac{T}{R} \quad (R \leqslant r < \infty). \tag{18}$$

where the subscripts 1 and 2 denote respectively the quantities prescribed in the inner and outer regions separated by the vortex sheet, and R_1 is a reference radial location in the inner region. Let the vortex sheet be perturbed according to (14). The total pressure should be balanced at the perturbed interface, i.e.

$$Q_1(R+\hat{\eta}) = Q_2(R+\hat{\eta})+T\left(\frac{1}{R_{r\theta}}+\frac{1}{R_{rz}}\right), \tag{19}$$

where $R_{r\theta}$ and R_{rz} are the principal curvatures of the disturbed surface. Subtracting (17) and (18) from (19) and assuming that all the quantities in the mean flow are bounded and continuous in the interval $[R, R+\hat{\eta}]$, we obtain the first-order perturbation condition for dynamical balance at the undeformed interface as follows:

$$\langle\hat{q}\rangle+\left[\left\langle\rho_0\left(\frac{V_0^2}{r}-\frac{V_A^2}{r}\right)\right\rangle+T\left(\frac{\partial^2}{\partial z^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\right)\right]\hat{\eta}(R) = 0. \tag{20}$$

The dynamic interfacial condition (13) derived by the normal-mode analysis is recovered if periodic solutions for the perturbation quantities are assumed once again.

As shown in figure 1, the mathematical steps adopted to derive (20) from (19) simply demonstrate a dissolution of the total pressure force acting at the *disturbed* surface of the vortex sheet (figure 1*a*) into the individual force components acting at the *steady-state interface* (figure 1*b*). As a matter of fact, (20) can also be reached

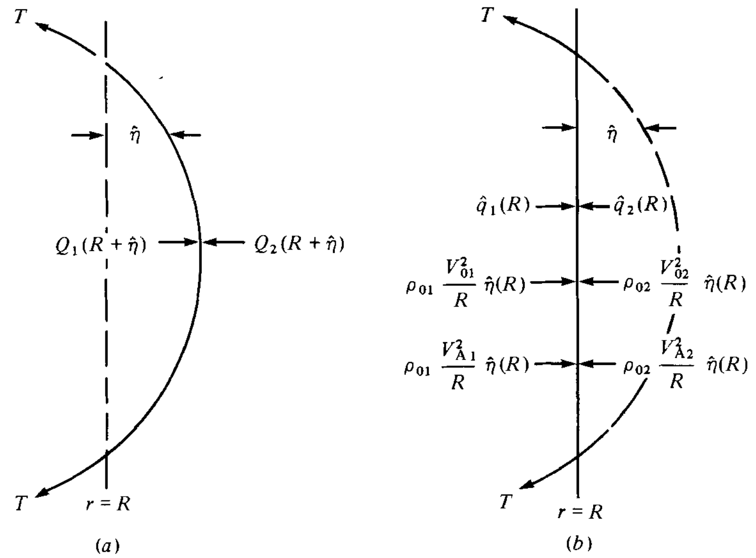

FIGURE 1. Dissolution of the total force into the force components at the interface.

simply by balancing all the force components acting at a differential element $\hat{\eta}_{(R)}R\,\mathrm{d}\theta$ (per unit axial wavelength) that experiences the centripetal acceleration induced by the angular velocity and magnetic flux. This procedure of force decomposition demonstrates that the deformation of the interface affects the flow in two ways: perturbing the pressure field (including the magnetic pressure) and disturbing the centrifugal force field (generated by the azimuthal velocity and magnetic flux).

The derivation just given clearly shows that (15) and (20) are respectively purely kinematic and dynamic conditions. They are therefore valid for both compressible and incompressible flows, and accordingly, compressibility effects do not explicitly enter conditions (12) and (13) derived in §2.

4. A general class of vortex sheets

The centrifugal force enters the dynamic interfacial condition through its interaction with the Lagrangian displacement in the r-direction. The interaction represents the influence on flow stability due to the perturbation to the centrifugal force field. To understand such an influence, we will analyse a general class of vortex sheets subject to different perturbations. For simplicity, the perturbations are restricted to be temporal, i.e. the axial wavenumber k is real while the frequency $\omega = \omega_r + \mathrm{i}\omega_i$ is complex. Compressibility effects are ignored. The governing stability equations reduced from (9) and (10) under the present assumptions are given as

$$(N^2-N_A^2)\,\mathrm{D}^*\left(\frac{u}{N}\right) - \frac{2m}{r}(N\Omega-N_A\Omega_A)\left(\frac{u}{N}\right) = \frac{\mathrm{i}}{\rho_0}\left(k^2+\frac{m^2}{r^2}\right)q, \tag{21}$$

$$\left\{(N^2-N_A^2)\left[(N^2-\Phi)-(N_A^2-\Psi_A)\right]-4\Omega^2N_A^2\left[\left(1-\frac{\Omega_A}{\Omega}\right)^2+2\frac{\Omega_A}{\Omega}\left(1-\frac{N}{N_A}\right)\right]\right\}\left(\frac{u}{N}\right)$$
$$= \frac{\mathrm{i}}{\rho_0}\left[(N^2-N_A^2)\,\mathrm{D}q+\frac{2m}{r}(N\Omega-N_A\Omega_A)\,q\right], \tag{22}$$

where $\Omega = V_0/r$ is the angular velocity and $\Omega_A = V_A/r$ is the Alfvén angular velocity.

For the convenience of mathematical operations and discussion, we define

$$F_c = \langle \rho_0 r(\Omega^2 - \Omega_A^2)\rangle + \frac{T}{R^2}(\kappa^2 + m^2 - 1), \tag{23}$$

with $\kappa = kR$. The interfacial condition (13), now written as

$$\langle q\rangle - \mathrm{i}\left(\frac{u}{N}\right)_R F_c = 0, \tag{24}$$

can then be viewed as the dynamical balance condition between the perturbation of the total pressure (including the magnetic pressure) and the perturbation of the unbalanced centrifugal forces and surface tension at the interface.

The general class of vortex-sheet profiles to be considered has two flow regions with their steady-state interface located at $r = R$. The flow properties in the inner region are

$$\left.\begin{aligned} \rho_0(r) = \rho_1, \quad \Omega(r) = \Omega_1, \quad & W(r) = W_1, \\ \Omega_A(r) = \Omega_{A1}, \quad & W_A(r) = W_{A1} \end{aligned}\right\} \quad (0 \leqslant r < R), \tag{25}$$

where the quantities with numerical indices are constant. The flow properties in the outer region are arbitrary functions of the radius. By applying the boundary condition at the axis the solution for the perturbation velocity u_1 in the inner region can be obtained from (21) and (22) as

$$u_1 = AN_1\left\{\frac{2m(N_1\Omega_1 - N_{A1}\Omega_{A1})}{r(N_1^2 - N_{A1}^2)} + \frac{kg_1 I_m'(kg_1 r)}{I_m(kg_1 r)}\right\} I_m(kg_1 r), \tag{26}$$

where

$$g_1 = \left\{1 - 4\left(\frac{N_1\Omega_1 - N_{A1}\Omega_{A1}}{N_1^2 - N_{A1}^2}\right)^2\right\}^{\frac{1}{2}},$$

$$N_1 = kW_1 + m\Omega_1 - \omega,$$

$$N_{A1} = kW_{A1} + m\Omega_{A1},$$

and $I_m(kg_1 r)$ is the modified Bessel function of the first kind. The prime denotes the total derivative with respect to the argument of the Bessel function. The solution in the inner region will be used to analyse the influence of the centrifugal force field on flow stability subject to three kinds of disturbance at the interface: an axisymmetric perturbation, an azimuthal perturbation, and an arbitrary perturbation.

Case 1: *the axisymmetric mode* ($m = 0$)

The solution in the inner region as described by (26) for the axisymmetric case reduces to

$$u_1 = AN_1 kg_1 I_0'(kg_1 r), \tag{27}$$

where

$$N_1 = kW_1 - \omega,$$

$$g_1 = \left\{1 - 4\left(\frac{N_1\Omega_1 - kW_{A1}\Omega_{A1}}{N^2 - k^2 W_{A1}^2}\right)^2\right\}^{\frac{1}{2}}.$$

For the convenience of mathematical operations, we will express the dynamical interfacial condition only in terms of the perturbation velocity. To do this, we substitute (21) into (24) for $m = 0$ and obtain

$$\left\langle \rho_0 r^2(N^2 - k^2 W_A^2)\,\mathrm{D}^*\left(\frac{u}{N}\right)\right\rangle + \kappa^2\left(\frac{u}{N}\right) F_c = 0. \tag{28}$$

The governing stability equation obtained by combining (21) and (22) for the axisymmetric mode reduces to the form

$$\mathrm{D}\left[\rho_0(N^2-k^2W_A^2)\,\mathrm{D}^*\left(\frac{u}{N}\right)\right]-\rho_0 k^2[(N^2-k^2W_A^2)g^2+4\Omega^2-\Phi+\Psi_A]\left(\frac{u}{N}\right)=0, \quad (29)$$

where

$$g^2 = 1-4\left(\frac{N\Omega-kW_A\Omega_A}{N^2-k^2W_A^2}\right)^2 \quad (R \leqslant r < \infty).$$

Multiplying (29) by $r(\bar{u}/\bar{N})$, where the quantities with a bar represent the complex conjugates, and integrating the resultant equation over the outer region, we obtain, after combining (27) with (28) and applying the boundary condition at infinity, the following integral equation:

$$k^2\kappa I_0'(\kappa\bar{g}_1)\{\rho_1 g_1 I_0(\kappa g_1)(N_1^2-k^2W_{A1}^2)-\kappa I_0'(\kappa g_1)F_c R\}$$
$$+\int_R^\infty \rho_0(N^2-k^2W_A^2)[|\mathrm{D}^*\phi_k|^2+k^2g^2|\phi_k|^2]\,r\,\mathrm{d}r-\int_R^\infty \rho_0 k^2(\Phi-4\Omega^2-\Psi_A)|\phi_k|^2\,r\,\mathrm{d}r=0. \quad (30)$$

Here the transformation $\phi_k = (1/Ag_1)\,u/N$ has been applied for $R \leqslant r < \infty$.

To observe the effects of the centrifugal force field and other flow quantities on the stability of the flow and especially of the interface, the following special profiles are considered to simplify the integral equation (30). Let $\Omega_1 = 0$ in the inner region and $W_A = 0$ throughout the entire flow field. After some mathematical manipulations, we have found that the flow under the present assumptions will be stable when

$$-\left\{\alpha_k\int_R^\infty (W-W_1)^2X_k\,\mathrm{d}r+\delta_k\right\}$$
$$+\left(\alpha_k+\int_R^\infty X_k\,\mathrm{d}r\right)\left\{\int_R^\infty \rho_0(\Phi-\Psi_A)|\phi_k|^2\,r\,\mathrm{d}r+[\kappa I_0'(\kappa)]^2F_c\right\} \geqslant 0, \quad (31)$$

where

$$\alpha_k = k^2\rho_1\kappa I_0'(\kappa)I_0(\kappa) \geqslant 0,$$
$$X_k = \rho_0(|\mathrm{D}^*\phi_k|^2+k^2|\phi_k|^2)\,r \geqslant 0,$$
$$\delta_k = \int X_k\,\mathrm{d}r\int W^2X_k\,\mathrm{d}r-\left(\int WX_k\,\mathrm{d}r\right)^2.$$

The first and second pairs of curly brackets in (31) respectively contain information on the axial velocities and on the centrifugal forces. Since both α_k and X_k are positive-definite, the first integral in the first pair of curly brackets represents the axial-velocity difference at the interface, always destabilizing the flow. The second term in the brackets contains information on the axial velocity in the outer region. From the Schwarz inequality that $\delta_k \geqslant 0$ for all values of W, we can conclude that the presence of the axial velocity in the outer region always destabilizes the flow except for constant values of axial flows where $\delta_k \equiv 0$. The tangential shears at the interface and in the outer region are suppressed since the perturbations are only axisymmetric. The first term in the second pair of curly brackets is the integral of the Rayleigh–Synge and the Alfvén discriminants. For vortex flows subject to axisymmetric disturbances and in the absence of the axial velocity and the axial magnetic field, the two discriminants constitute the generalized Michael condition to represent a state of centrifugal stability as described in (11). The corresponding integral in the curly brackets therefore conveys information on centrifugal stability in the outer region. The last term in the second pair of curly brackets carries the information on the centrifugal forces acting on both sides of the interface, parallel to the first term in the same brackets. As pointed out in the derivation of the dynamic interfacial condition in (13), the jump condition arising from the perturbation of the centrifugal force field is the outcome of integrating the Rayleigh–Synge and the Alfvén discriminants across the interface. The last term in the brackets can then be viewed as the integral representation of the generalized Michael condition at the interface. The sign of F_c, indicating whether or not the resultant force at the interface is centrifugally stable, determines the stabilizing or destabilizing effect on the flow. In the present case

$$F_c = R[\rho_2 \Omega_2^2 - (\rho_2 \Omega_{A2}^2 - \rho_1 \Omega_{A1}^2)] + \frac{T}{R^2}(\kappa^2 - 1). \tag{32}$$

The surface tension always stabilizes the flow except for very long axial wavelengths where $\kappa < 1$. The presence of the magnetic field in the inner region stabilizes the flow while that in the outer region destabilizes the flow. The rotational velocity immediately outside the sheet always stabilizes the flow.

It should be pointed out that the flow profile being considered in this axisymmetric case can be reduced to the one examined by Leibovich (1969) if the axial flow in the outer region and all the magnetic forces in the flow field are deleted. Equations (31) with $W = \Psi_A = 0$ should have been recovered had the correct interfacial condition described by (24) been used in his paper. To re-evaluate his conclusions, we follow the same procedures in his paper and discover that the flow will be stable if the following two conditions are met:

$$\int \rho_0(\Phi - \Psi_A)\,|\phi_k|^2 r\,\mathrm{d}r + [\kappa I_0'(\kappa)]^2 F_c \geqslant \alpha_k \frac{\int (W - W_1)^2 X_k\,\mathrm{d}r}{\int X_k\,\mathrm{d}r} + \frac{\delta_k}{\alpha_k + \int X_k\,\mathrm{d}r}, \tag{33}$$

$$\int \rho_0[(\Phi - \Psi_A) - k^2(W - W_1)^2]\,|\phi_k|^2 r\,\mathrm{d}r + [\kappa I_0'(\kappa)]^2 F_c \geqslant \int (W - W_1)^2 \rho_0 |\mathrm{D}^*\phi_k|^2 r\,\mathrm{d}r + \frac{\delta_k}{\alpha_k + \int X_k\,\mathrm{d}r}. \tag{34}$$

Both (33) and (34) reduce to the generalized Michael condition in the outer region and at the interface if no axial velocity gradient exists throughout the flow field. A conclusion that the flow will be unstable for large k could immediately be drawn from the above two equations if F_c were neglected regardless of the centrifugal-force balance condition at the interface. Such a negligence led him to conclude that the flow must be unstable at least to short waves and possibly to all wavelengths. While flows of the vortex-sheet type are susceptible to short-wave perturbations because of the strong shear effect present at the interface, the centrifugal force jump as in the case investigated by Leibovich (1979) will certainly stabilize those perturbations with longer wavelengths. This characteristic is clearly shown in (31) and will be supported by an exact solution to be given in the following.

Because of the presence of the centrifugal term involving F_c in (31), instabilities for large axial wavenumber cannot immediately be concluded. Therefore it is necessary to obtain solutions for some specific flow profiles in the outer region before

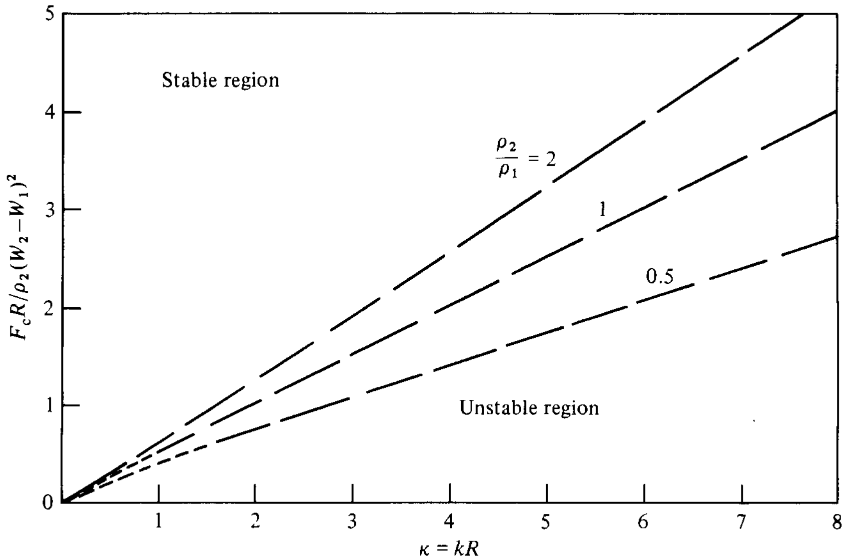

FIGURE 2. Stability domains for flows subject to axisymmetric perturbations.

the detailed stability phenomenon can be observed. We will examine the following flow profile:

$$\left.\begin{array}{ll}\rho_0(r) = \rho_2, & W(r) = W_2, \\ \Omega(r) = \Omega_2(R/r)^2, & \Omega_A(r) = \Omega_{A2}\end{array}\right\} \quad (R \leqslant r < \infty), \tag{35}$$

where ρ_2, W_2, Ω_2, and Ω_{A2} are constants. The perturbation velocity u_2 in the outer region obtained by solving (29) is

$$u_2 = B(kW_2 - \omega)\, kK_0'(kr).$$

The stability boundary described by (31) in this case is

$$-k^2(W_1 - W_2)^2 + \left[\frac{1}{\rho_1}\frac{\kappa I_0'(\kappa)}{I_0(\kappa)} - \frac{1}{\rho_2}\frac{\kappa K_0'(\kappa)}{K_0(\kappa)}\right]\frac{F_c}{R} \geqslant 0, \tag{36}$$

where F_c is given in (32). Since the sum of the terms within the square brackets are positive, the stabilizing effect, if any, will come from the centrifugal force term F_c. Even though (36) may still be violated for very large wavenumbers, perturbations corresponding to smaller axial wavenumbers will certainly be stabilized by the centrifugally stable forces at the interface. As demonstrated by (36), an erroneous conclusion that no axisymmetric modes are stable can easily be reached if the perturbation of the centrifugal force at the interface is omitted. Figure 2 shows such stability domains for the ratio between the centrifugal-force and the axial-velocity difference as a function of the axial wavenumber. The centrifugal-force jump at the interface does stabilize perturbations with smaller axial wavenumbers.

Case 2: the azimuthal mode ($k = 0$)

The solutions in the inner region governed by (26) for azimuthal modes have the forms

$$u_1 = AN_1 r^{m-1}, \tag{37}$$

where

$$N_1 = m\Omega_1 - \omega,$$

and m will be treated as a positive integer. We will again express the dynamic interfacial condition only in terms of the perturbation velocity. Combining (21) and (24) for $k = 0$ yields

$$\left\langle \rho_0 r^2(N^2 - m^2\Omega_A^2)\, D^*\left(\frac{u}{N}\right)\right\rangle + \left(\frac{u}{N}\right)_R \{-\langle 2mr\rho_0(N\Omega - m\Omega_A^2)\rangle + m^2 F_c\} = 0. \quad (38)$$

As for the outer region, the governing equation obtained from (21) and (22) for azimuthal modes is

$$D\left[\rho_0 r^2(N^2 - m^2\Omega_A^2)\, D^*\left(\frac{u}{N}\right)\right]$$
$$-\{2mr\, D[\rho_0(N\Omega - m\Omega_A^2)] + \rho_0 m^2(N^2 - \Phi + 4\Omega^2 - m^2\Omega_A^2 + \Psi_A\}\left(\frac{u}{N}\right) = 0 \quad (R \leqslant r < \infty). \quad (39)$$

Following the procedures used in case 1, we can easily conclude that again the sign of F_c determines the stabilizing or destabilizing effect on the flow. In the present case

$$F_c = R[(\rho_2\Omega_2^2 - \rho_1\Omega_1^2) - (\rho_2\Omega_{A2}^2 - \rho_1\Omega_{A1}^2)] + \frac{T}{R^2}(m^2-1), \quad (40)$$

represents the balanced or unbalanced force at the interface. The stability condition for this azimuthal case is found to be

$$x_1 + x_2 + x_3 \geqslant 0, \quad (41)$$

where

$$x_1 = -\left\{(m-1)\rho_1 \int (\Omega - \Omega_1)^2\, Y_m\, dr + \delta_m\right\},$$

$$x_2 = \left[\rho_2 - \rho_1 + \int (D\rho_0)\, r^2|u|^2\, dr\right]\left[(m-1)\rho_1\Omega_1^2 + \int \Omega^2 Y_m\, dr\right],$$

$$x_3 = \left(m\rho_1 + \int X_m\, dr\right)\left[(m^2-1)\frac{T}{R^3} + (m-1)\rho_1\Omega_{A1}^2 + \int \Omega_A^2 X_m\, dr\right],$$

$$X_m = \rho_0(r^2|D^*\phi_m|^2 + m^2|\phi_m|^2)\, r \geqslant 0,$$

$$Y_m = \rho_0[r^2|D\phi_m|^2 + (m^2-1)\,|\phi_m|^2]\, r \geqslant 0,$$

$$\phi_m = \frac{1}{AR^{2m}}\frac{u}{N},$$

$$\delta_m = \int Y_m\, dr \int \Omega^2 Y_m\, dr - \left(\int \Omega Y_m\, dr\right)^2.$$

The roles played by the flow quantities on the stability mechanism can be observed by examining (41). Again from the Schwarz inequality that $\delta_m \geqslant 0$ for all values of Ω, the terms in x_1 in (41) always destabilize the flow. Furthermore, by comparing the terms in x_1 with those in the first pair of curly brackets in (31), we can draw an analogy between the two and conclude that they both convey shear effects which destabilize the flow. The difference is that the shear effect in the present case is generated by the angular-velocity gradient rather than the axial-velocity gradient. The term x_2 is the contribution to stability by the density variation in the centrifugal force field. Obviously the stabilizing or destabilizing effect depends on the density difference at the interface and on the density distribution in the outer region. Densities that increase with radius always stabilize the flow and *vice versa*, as one

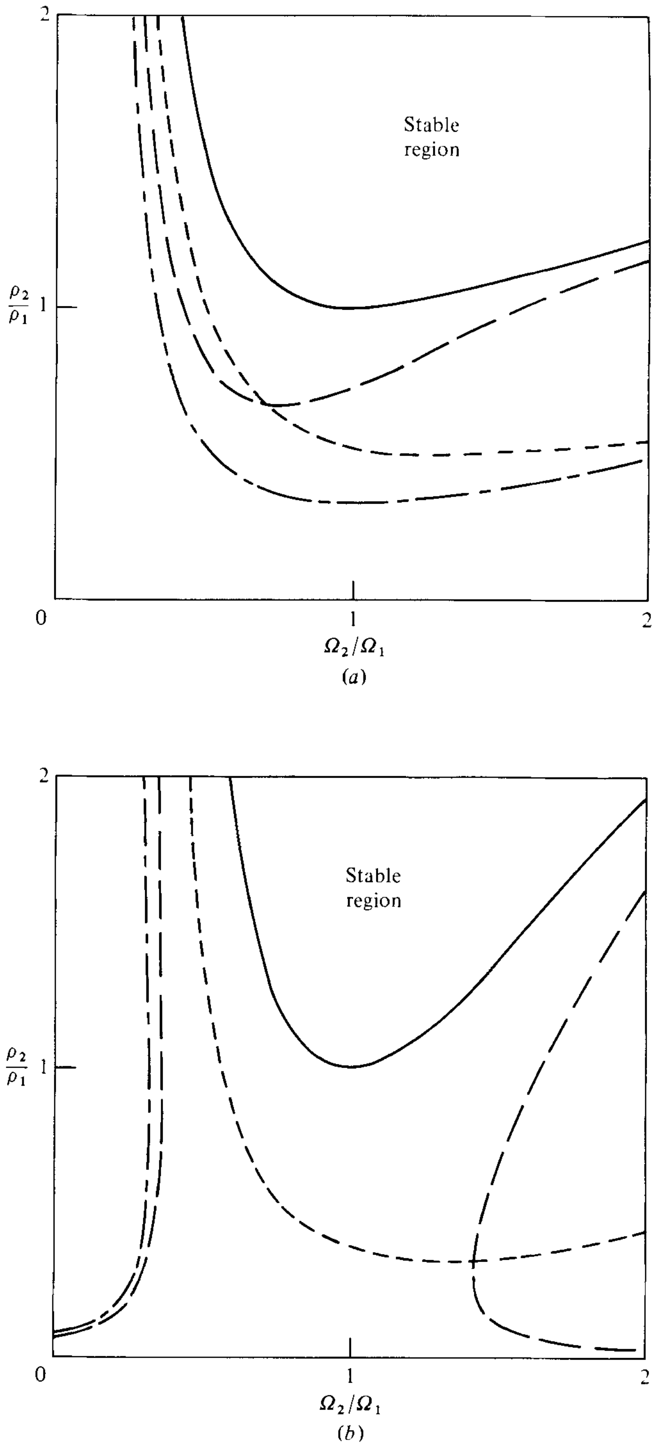

FIGURE 3(a, b). For caption see facing page.

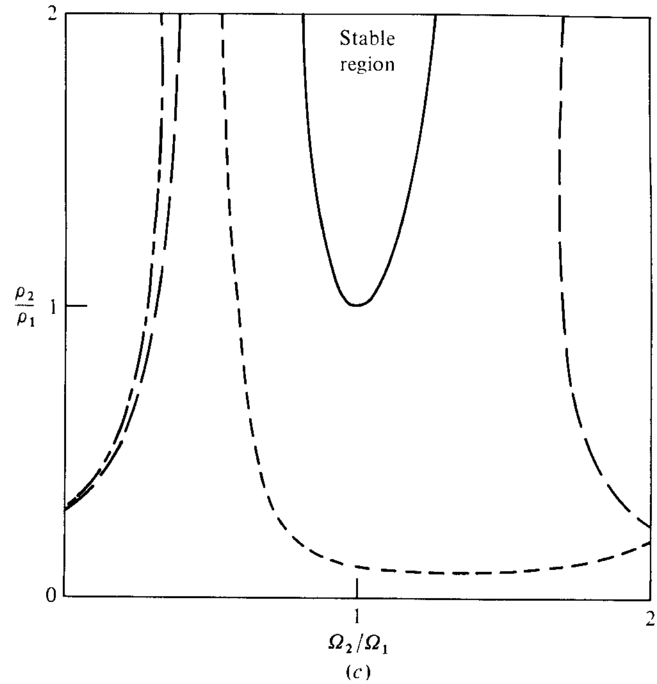

FIGURE 3. Stability domains for flows subject to $m = 2(a)$, $5(b)$ and $30(c)$ azimuthal perturbations: ——, $\Omega_{A1} = \Omega_{A2} = 0$; – – –, $\Omega_{A1}/\Omega_1 = 0.5$; ----, $\Omega_{A1} = 0$, $\Omega_{A2}/\Omega_2 = 0.5$; —·—·—, $\Omega_{A1}/\Omega_1 = \Omega_{A2}/\Omega_2 = 0.5$.

would intuitively expect. The above discussion reveals that the angular velocity in rotating flows plays a dual role in flow stabilities: the velocity gradient produces a shear effect while the velocity itself induces a centrifugal force field. The term x_3 contains the information on the surface tension and on the magnetic field. The surface tension always stabilizes non-axisymmetric perturbations as is well known. The azimuthal magnetic fields in both the inner and outer region always stabilize the flow in spite of the details of the magnetic profile. This characteristic is also true for arbitrary flows if only the perturbations in the azimuthal direction are permitted.

To further illustrate the stability characteristics described by (41), we consider a special flow profile

$$\rho_0(r) = \rho_2, \quad \Omega(r) = \Omega_2, \quad \Omega_A(r) = \Omega_{A2} \quad (R \leqslant r < \infty). \tag{42}$$

All the quantities with numerical indices are constants. The solution, obtained by solving (39) for the present flow profile, is

$$u_2 = BmN_2\left[\frac{2(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} - 1\right]r^{-m-1}. \tag{43}$$

The secular relation for stability is

$$(\rho_1+\rho_2)\,\omega^2 - 2[(m-1)\,\rho_1\,\Omega_1 + (m+1)\,\rho_2\,\Omega_2]\,\omega + m\left[(m-2)\,\rho_1(\Omega_1^2 - \Omega_{A1}^2) + (m+2)\,\rho_2(\Omega_2^2 - \Omega_{A2}^2) - \frac{F_c}{R}\right] = 0. \tag{44}$$

As previously predicted, we can immediately conclude that the sign of F_c determines whether or not the force condition at the interface stabilizes the flow. For stability ($\omega_i = 0$) the characteristic equation from (44) requires that

$$-(m^2-1)\rho_1\rho_2(\Omega_1-\Omega_2)^2+(\rho_2-\rho_1)[(m-1)\rho_1\Omega_1^2+(m+1)\rho_2\Omega_2^2] + m(\rho_1+\rho_2)\left[(m-1)\rho_1\Omega_{A1}^2+(m+1)\rho_2\Omega_{A2}^2+(m^2-1)\frac{T}{R^3}\right] \geqslant 0. \quad (45)$$

As previously discussed, the first term in the above equation represents the shear effect generated by the angular velocity gradient at the interface, always destabilizing the flow. Because of the uniform rotation for $r \geqslant R$ implying $\delta_m \equiv 0$, no shear effect exists in the outer region. The second term is the effect of the density difference experienced in the centrifugal forces generated at the interface. Stabilizing effects correspond to larger density in the outer region. The last term contains the information on the surface tension and on the azimuthal magnetic fields in the inner and outer region, all always stabilizing azimuthal perturbations. Figures 3(a, b, c) show the stability domains for $m = 2$, 5 and 30 modes in the absence of surface tension. Both the destabilizing effect produced by the shear at the interface and the stabilizing effect induced by the azimuthal magnetic fields increase as the wavenumber becomes larger. For very large m, (45) for zero surface tension reduces to

$$-\rho_1\rho_2(\Omega_1-\Omega_2)^2+(\rho_1+\rho_2)(\rho_1\Omega_{A1}^2+\rho_2\Omega_{A2}^2) \geqslant 0. \quad (46)$$

The flow in the absence of the magnetic force is therefore always unstable except for uniform rotation where $\Omega_1 = \Omega_2$.

Case 3: the arbitrary mode

For simplicity consider $\Omega_1 = \Omega_{A1} = 0$ in the inner region, and the solution for the perturbation velocity reduced from (26) is

$$u_1 = AN_1 kI'_m(kr). \quad (47)$$

The dynamic interfacial condition expressed in terms of the perturbation velocities from (21) and (24) is

$$\left\langle \rho_0 r^2(N^2-N_A^2)\,\mathrm{D}^*\left(\frac{u}{N}\right)\right\rangle + \left(\frac{u}{N}\right)_R \{\langle -2mr\rho_0(N\Omega-N_A\Omega_A)\rangle + (\kappa^2+m^2)F_c\} = 0. \quad (48)$$

For the outer region, when we consider $\Omega_A = W_A = 0$, the governing equation for stability is

$$\mathrm{D}\left[\frac{\rho_0 r^2 N^2}{m^2+k^2r^2}\mathrm{D}^*\left(\frac{u}{N}\right)\right] - \left\{2mr\,\mathrm{D}\left(\frac{\rho_0\Omega N}{m^2+k^2r^2}\right)+\rho_0\left(N^2-\Phi+\frac{4m^2\Omega^2}{m^2+k^2r^2}\right)\right\}\left(\frac{u}{N}\right) = 0. \quad (49)$$

Following the procedures used in the axisymmetric case, we can again conclude that the sign of F_c determines the stability effect carried by the forces acting at the interface. In the present case

$$F_c = \rho_2 R\Omega_2^2 + \frac{T(\kappa^2+m^2-1)}{R^2}, \quad (50)$$

is positive-definite. The surface tension always stabilizes non-axisymmetric perturbations as is well known. The centrifugal-force term arising from the difference in the angular velocity at the interface always stabilizes the flow. The stability condition is found to be

$$y_1+y_2+y_3+y_4 \geqslant 0, \tag{51}$$

where

$$y_1 = -\alpha\int_R^\infty k^2(W_1-W)^2\,X\,\mathrm{d}r-\delta_W-\delta_\Omega,$$

$$y_2 = 2\alpha\int_R^\infty k(W_1-W)\,m\Omega\,Y\,\mathrm{d}r$$

$$+2\left\{\int_R^\infty kWX\,\mathrm{d}r\int_R^\infty m\Omega\,Y\,\mathrm{d}r-\int_R^\infty X\,\mathrm{d}r\int_R^\infty kWm\Omega\,Y\,\mathrm{d}r\right\},$$

$$y_3 = \left\{\kappa I'_m(\kappa)\left[\rho_2\frac{\kappa I'_m(\kappa)}{m^2+\kappa^2}-\rho_1 I_m(\kappa)\right]+\int_R^\infty \mathrm{D}\left(\frac{\rho_0}{m^2+k^2r^2}\right)r^2|\phi|^2\,\mathrm{d}r\right\}\int_R^\infty m^2\Omega^2\,Y\,\mathrm{d}r$$

$$+\left(\alpha+\int X\,\mathrm{d}r\right)\int\left(\Phi+\frac{2m^2\Omega^2}{m^2+k^2r^2}\right)\frac{k^2r^2}{m^2+k^2r^2}|\phi|^2\,r\,\mathrm{d}r,$$

$$y_4 = \left(\alpha+\int X\,\mathrm{d}r\right)\left\{\alpha k^2W_{\mathrm{A}1}^2+[\kappa I'_m(\kappa)]^2\left[\frac{\kappa^2\rho_2\Omega_2^2}{m^2+\kappa^2}+\frac{T}{R\kappa^2}(\kappa^2+m^2-1)\right]\right\},$$

$$X = \rho_0\left[\frac{r^2}{m^2+k^2r^2}|\mathrm{D}^*\phi|^2+|\phi|^2\right]r \geqslant 0,$$

$$Y = \frac{\rho_0}{m^2+k^2r^2}[r^2|\mathrm{D}\phi|^2+(k^2r^2+m^2-1)\,|\phi|^2]\,r \geqslant 0,$$

$$\alpha = \rho_1\,\kappa I'_m(\kappa)\,I_m(\kappa) \geqslant 0,$$

$$\phi = \frac{1}{A}\frac{u}{N},$$

$$\delta_W = \int X\,\mathrm{d}r\int k^2W^2X\,\mathrm{d}r-\left(\int kWX\,\mathrm{d}r\right)^2,$$

$$\delta_\Omega = \int Y\,\mathrm{d}r\int m^2\Omega^2\,Y\,\mathrm{d}r-\left(\int m\Omega\,Y\,\mathrm{d}r\right)^2.$$

From the Schwarz inequality, it follows that $\delta_W \geqslant 0$ and $\delta_\Omega \geqslant 0$ for all values of W and Ω. Several stability characteristics can then be observed from (51) as follows.

The quality y_1 carries the shear effects in both axial and azimuthal directions, always destabilizing the flow. The first term in y_1 is the axial shear generated by the velocity difference between the inner and the outer regions. The second term δ_W and the third term δ_Ω are respectively the shear effects produced by the axial- and azimuthal-velocity differences within the outer region. The corresponding terms can be found in the case for the axisymmetric mode and for the azimuthal mode.

The quantity y_2 is the shear-effect interaction between the velocities in the axial and azimuthal directions. The first term in y_2 is the interaction between the inner and outer regions, while the second term in the interaction within the outer region. All the terms in y_2 can be positive or negative, depending on the signs of the velocities and of the wavenumbers, and therefore can stabilize or destabilize the flow. Such dependence implies whether or not the axial or azimuthal shear reinforces each other and whether or not the direction of perturbations strengthens the resultant shear effect.

The first term in y_3 is the effect of density variations at the interface and in the outer region in the centrifugal force field created by the rotation of the fluid. Densities increasing radially outwards stabilize the flow. The second term in y_3 involves the integration of the Rayleigh–Synge discriminant over the outer region. The condition for centrifugally stable profiles, i.e. $\Phi \geqslant 0$, is the precondition for the sufficiency condition of stability for flows subject to perturbations in both the axial and azimuthal directions (Fung & Kurzweg 1975). Positive values of Φ stabilize the flow.

The quantity y_4 carries the information on the forces acting at the interface. The presence of the surface tension and of the axial magnetic field in the inner region always stabilize the flow. The term involving $\rho_2 \Omega_2^2$ in y_4 is the centrifugal force created by the rotation of the outer region at the interface, always stabilizing the flow.

To further illustrate the stability characteristics described by (51), we consider the following profile:

$$\rho_0(r) = \rho_2, \quad W(r) = W_2, \quad \Omega(r) = \Omega_2\left(\frac{R}{r}\right)^2 \quad (R \leqslant r < \infty). \tag{52}$$

Here ρ_2, W_2 and Ω_2 are constant. The solution for the perturbation velocity obtained from (49) for the present flow profile is

$$u_2 = B(kW_2 + m\Omega_2 - \omega)\, kK_m'(kr). \tag{53}$$

Equation (51) then has the form

$$-\rho_1\rho_2[k(W_1 - W_2) - m\Omega_2]^2 + \left[\rho_2 \frac{\kappa I_m'(\kappa)}{I_m(\kappa)} - \rho_1 \frac{\kappa K_m'(\kappa)}{K_m(\kappa)}\right]\left[\rho_1 k^2 W_{A1}^2 \frac{I_m(\kappa)}{\kappa I_m'(\kappa)} + \rho_2 \Omega_2^2 + \frac{T}{R^3}(\kappa^2 + m^2 - 1)\right] \geqslant 0. \tag{54}$$

The first term in (54) is the shear effect created by the velocity difference at the interface. The axial velocity difference always destabilizes the flow. However, such destabilization interacts with the radial shear effect generated by the rotation of the outer region. Whether or not the interaction reinforces the destabilization depends on the direction of the axial and azimuthal velocities and of the axial and azimuthal perturbations. Since $\kappa I_m'(\kappa)/I_m(\kappa) \geqslant 0$ and $\kappa K_m'(\kappa)/K_m(\kappa) \leqslant 0$, the second term in (54) is always positive and is contributed by the axial magnetic field in the inner region, the rotation of fluid in the outer region, and the surface tension at the interface. All these contributions stabilize the flow. It was the contribution from the unbalanced centrifugal force $\rho_2 \Omega_2^2$ neglected by Michalke & Timme (1967) in their analysis of inviscid instability of vortex-sheet flows with constant density. Such negligence led them to conclude that some of the perturbations which should have been stabilized by the unbalanced centrifugal forces at the vortex sheet were unstable.

5. Conclusions

The interfacial conditions for a cylindrical vortex sheet or fluid layer with radius-dependent density, velocity and magnetic fields have been obtained for isentropic compressible flows subjected to arbitrary spatial and temporal disturbances. The conditions are valid for vortex flows with or without discontinuities. The derivation of the dynamic interfacial conditions shows that the deformation of the vortex sheet affects the flow in two ways: disturbing the total pressure field and perturbing the centrifugal force field created by the azimuthal components of the velocity and the magnetic flux. The latter seems to be straightforward but is easily overlooked. Failure to consider such a perturbation to a stable centrifugal force at the vortex sheet can lead to the erroneous destabilization of certain modes corresponding to smaller axial and azimuthal wavenumbers.

The present analysis has also demonstrated several characteristics of a general class of vortex flows. Unlike the velocity in the two-dimensional parallel flows, the rotation of vortex motions plays a dual role in flow stability: the angular-velocity gradient produces shear effects which always destabilize the flow while the angular velocity itself generates a centrifugal force field which can stabilize or destabilize the flow. The stabilization or destabilization depends on whether the force field is centrifugally stable or unstable. For flows of the vortex-sheet type, the centrifugal force arising from the discontinuities in the rotating velocity and the azimuthal magnetic field at the vortex sheet has significant influence on flow stability. The resultant direction of the centrifugal force at the interface, dictated by the sign of F_c in (23), determines whether such force stabilizes or destabilizes the flow. As shown in (24), the forces acting at both sides of the vortex sheet interact with the perturbation displacement of the deformed interface. Such an interaction will stabilize perturbations corresponding to smaller wavenumbers as shown in the examples in the present analysis. The azimuthal magnetic field always stabilizes azimuthal perturbations while the surface tension stabilizes all perturbations except for axisymmetric ones with long axial wavelengths.

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